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# CONTROLLABLE MOTIONS OF A TWO-LINK MECHANISM ALONG A HORIZONTAL PLANE†

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Controllable two-dimensional motions of a two-link mechanism along a horizontal plane are considered. Forces of dry friction act between the mechanism and the plane. The control is accomplished through a controlling torque, developed by a motor whose axis coincides with the axis of the hinge in the mechanism. Periodic control laws are constructed which guarantee longitudinal displacement of the two-link mechanism as a whole. The velocity of these motions is estimated. It is shown that any prescribed displacement of the mechanism in the plane may be achieved, and the corresponding motion is demonstrated. The motions constructed may serve as a simplified model for the motion of snakes and other animals with no extremities. The proposed mode of displacement may be used in the design of mobile robots, in particular, for robots of small dimensions. © 2001 Elsevier Science Ltd. All rights reserved.

Various aspects of the mechanics of snakes have been discussed (see, e.g. [1–4]), as have questions of the mechanics of robots employing the principles of snake locomotion; it is generally assumed that a snake is capable of utilizing surface irregularities, vertical walls, stones, grass, etc., by pushing against which it produces a horizontal component of the normal reaction. This makes it possible to obtain a component of the force of friction directed forward along the snake's path. An analogous result is obtained in robot-engineering systems built of separate links equipped with wheels [4]. For these non-holonomic mechanical systems, support at side walls is replaced by a reaction normal to the plane of the wheels. However, it remained unclear how a wheel-less multi-link mechanism could move along a horizontal plane.

With reference to the example of a three-link mechanism, it has been shown [5, 6] that, by controlling the two inner torques applied at the hinges, one can make the mechanism move along a rough horizontal surface in any given direction. In the process, slow and fast phases of motion alternate. It has been shown [7] that a multi-link mechanism can be made to move using only slow (quasi-static) motions.

In this paper we investigate the motion of the simplest kind of multi-link mechanism – a two-link one consisting of two rigid bodies, equipped with a motor mounted in the hinge – along a rough horizontal surface. It will be shown that, through control by a single motor, the mechanism can be made to move forward and also be displaced arbitrarily in the horizontal plane. The displacements and velocities of motion will be estimated for different mechanical models of a tow-link mechanism.

## 1. MECHANICAL MODEL

A two-link mechanism consists of two absolutely rigid bodies connected by a cylindrical hinge (Fig. 1). Both bodies perform plane motion in a fixed horizontal plane, in such a way that the axis of the hinge is vertical. We introduce a fixed Cartesian system of coordinates $Oxyz$ with the $Oz$ axis pointing vertically upward and the $Oxy$ plane coinciding with the plane over which the mechanism is moving; $O^*$ is the point at which the hinge axis intersects the $Oxy$ plane.

We will use the following notation: $m_i$ masses of the rigid bodies, $J_i$ are their moments of inertia about the hinge axis, $C_i$ are the projections of their mass centres on the $Oxy$ plane and $a_i = O^*C_i$ are the distances from the hinge axis $O^*$ to the points $C_i$. The values of the subscript $i = 1, 2$ correspond to the two rigid bodies, which we shall call the body ($i = 1$) and the tail ($i = 2$), respectively. We shall assume that the hinge is a point mass $m_0$, which is not part of either of the two rigid bodies. This assumption reflects the real situation in which an electrical motor producing a controlling torque and possessing a considerable mass is mounted on the hinge axis. Thus, the total mass of the two-link mechanism is

$$m = m_0 + m_1 + m_2 \tag{1.1}$$

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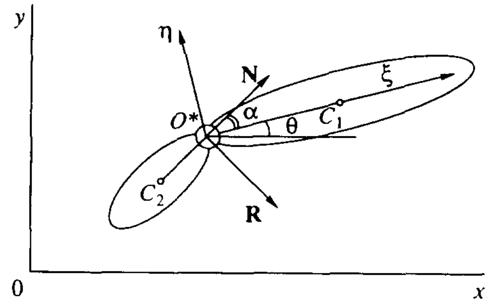


Fig. 1

The controlling torque produced by the motor about the hinge axis and applied to the tail will be denoted by $M$. A torque $-M$ then acts on the body.

Forces of dry friction obeying Coulomb's law act at the point where the mechanism is in contact with the plane. At moving points of contact, the frictional force is directed against the velocity of the point and has a magnitude $Pk$, where $P$ is the normal reaction at the point and $k$ is the constant coefficient of friction. For fixed points of contact, the friction force has a value not exceeding $Pk$ and may have any direction. The coefficients of friction for points of the body, tail and point mass $m_0$ are denoted by $k_1$, $k_2$ and $k_0$, respectively. The difficulty is that, when there are more than three points of contact, the normal reactions are not uniquely defined, because of static indeterminacy. Hence the friction forces, even at moving points, are also not uniquely defined. We will have to deal with this difficulty later.

Suppose $x_0, y_0$ are the Cartesian coordinates of the point $O^*$, $\theta$ is the angle at which the axis of the body $O^*C_1$ is inclined to the $Ox$ axis, and $\alpha$ the angle between the axis of the tail $C_2O^*$ and that of the body $O^*C_1$ (see Fig. 1). Then the coordinates of the points $C_1$ and $C_2$ may be written as

$$x_1 = x_0 + a_1\cos\theta, \quad y_1 = y_0 + a_1\sin\theta \tag{1.2}$$
$$x_2 = x_0 - a_2\cos(\theta + \alpha), \quad y_2 = y_0 - a_2\sin(\theta + \alpha)$$

We now write down the coordinates $x_c, y_c$ of the mass centre $C$, which will be needed below

$$mx_c = m_1x_1 + m_2x_2 + m_0x_0 = mx_0 + m_1a_1\cos\theta - m_2a_2\cos(\theta + \alpha) \tag{1.3}$$
$$my_c = m_1y_1 + m_2y_2 + m_0y_0 = my_0 + m_1a_1\sin\theta - m_2a_2\sin(\theta + \alpha)$$

where we have used formulae (1.1) and (1.2).

Letting $\mathbf{v}_0$ denote the velocity vector of the point $O^*$, and $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ the angular velocity vectors of the body and tail, respectively, we can write the following expression for the angular momentum of the two-link mechanism about the point $O$

$$\mathbf{K} = m_0\overline{OO^*} \times \mathbf{v}_0 + m_1[\overline{OC_1} \times \mathbf{v}_0 + \overline{OO^*} \times (\boldsymbol{\omega}_1 \times \overline{O^*C_1})] +$$
$$+ m_2[\overline{OC_2} \times \mathbf{v}_0 + \overline{OO^*} \times (\boldsymbol{\omega}_2 \times \overline{O^*C_2})] + J_1\boldsymbol{\omega}_1 + J_2\boldsymbol{\omega}_2 =$$
$$= m\overline{OO^*} \times \mathbf{v}_0 + (m_1\overline{O^*C_1} + m_2\overline{O^*C_2}) \times \mathbf{v}_0 + [m_1(\overline{OO^*} \cdot \overline{O^*C_1}) + J_1]\boldsymbol{\omega}_1 +$$
$$+ [m_2(\overline{OO^*} \cdot \overline{O^*C_2}) + J_2]\boldsymbol{\omega}_2 \tag{1.4}$$

The vectors in (1.4) have the following components (see Fig. 1)

$$\overline{OO^*} = (x_0, y_0, 0), \quad \overline{O^*C_1} = (a_1\cos\theta, a_1\sin\theta, 0) \tag{1.5}$$
$$\overline{O^*C_2} = (-a_2\cos(\theta+\alpha), -a_2\sin(\theta+\alpha), 0)$$
$$\mathbf{v}_0 = (\dot{x}_0, \dot{y}_0, 0), \quad \boldsymbol{\omega}_1 = (0, 0, \dot{\theta}), \quad \boldsymbol{\omega}_2 = (0, 0, \dot{\theta} + \dot{\alpha})$$

Substituting formulae (1.5) into Eq. (1.4), we determine the magnitude of the angular momentum of the two-link mechanism in the form (the vector $\mathbf{K}$ is directed along the $Oz$ axis)

$$K = m(x_0\dot{y}_0 - y_0\dot{x}_0) + m_1a_1(x_0\dot{\theta}\cos\theta + y_0\dot{\theta}\sin\theta) -$$
$$-m_2a_2[x_0(\dot{\theta}+\dot{\alpha})\cos(\theta+\alpha) + y_0(\dot{\theta}+\dot{\alpha})\sin(\theta+\alpha)] + m_1a_1(\dot{y}_0\cos\theta - \dot{x}_0\sin\theta) -$$
$$-m_2a_2[\dot{y}_0\cos(\theta+\alpha) - \dot{x}_0\sin(\theta+\alpha)] + J_1\dot{\theta} + J_2(\dot{\theta}+\dot{\alpha}) \quad (1.6)$$

## 2. ELEMENTARY MOTIONS

The motions of the two-link mechanism will be constructed as a sequence of simple motions which we shall call "elementary motions" (EMs). All EMs begin from the mechanism's state of rest and end in a state of rest. The angle $\alpha$ between the body and the tail in each EM varies monotonically in the range $(-\pi, \pi)$ according to an arbitrary law $\alpha(t)$, which includes phases of acceleration and deceleration. We will denote the initial and final values of this angle in an EM by $\alpha^0$ and $\alpha^1$, respectively.

EMs subdivide into slow and fast motions. The duration of a slow motion will be denoted by $T$ and that of a fast one by $\tau$.

Slow motions are motions in which the body remains motionless while the tail turns through a certain angle. Below we will derive sufficient conditions under which slow motions may take place.

Fast motions are motions in which the magnitude $M$ of the controlling torque is much larger than the torques produced by the friction forces, while the duration $\tau$ of the motion is small. For fast motions we have

$$|M| \gg m'gka', \quad m' = \max(m_1, m_2), \quad a' = \max(a_1, a_2), \quad \tau \ll T \quad (2.1)$$

By virtue of conditions (2.1), friction forces can be neglected in the treatment of fast motions. Consequently, the laws of conservation of momentum and of angular momentum hold in such motions. At the beginning of the motion, however, the mechanism is in a state of rest; hence, for a fast motion

$$x_c \equiv \text{const}, \quad y_c \equiv \text{const}, \quad K \equiv 0 \quad (2.2)$$

Expressions for $x_c$, $y_c$ and $K$ are given by formulae (1.3) and (1.6).

## 3. ANALYSIS OF SLOW MOTIONS

To derive the conditions for the body to remain motionless in slow motions, the following scheme of reasoning will be used. First, assuming that the body is motionless, we will determine the forces with which the body and the tail interact, on the assumption that the tail is resting on the horizontal plane at points on the straight line $O^*C_2$. We will then consider the balance of forces and torques acting on the body and find the sufficient conditions under which the friction forces (in conditions of static indeterminacy) can balance out the other forces acting on the body (forces of interaction with the tail and controlling torque).

Let $N$ and $R$ denote the projections of the force, which the body applies to the tail, on to the direction of the segment $C_2O^*$ and on to the perpendicular direction (Fig. 1), respectively. We set up the equations of motion of the mass centre of the tail, projected on to these directions, on the assumption that the body is motionless:

$$m_2a_2\dot{\alpha}^2 = N, \quad m_2a_2\ddot{\alpha} = R - k_2\,\mathrm{sign}\,\dot{\alpha}\,\Sigma G_i \quad (3.1)$$

The last term is the sum of the friction forces acting on the tail and $G_i$ are the reactions normal to the $Oxy$ plane at the points of contact of the tail and the plane. The summation in (3.1) and in subsequent equations ranges over all points of contact. In the case of contact along an entire segment, the summation must be replaced by integration, but all other results remain unchanged.

The equation of the torques for the rotation of the tail about the point $O^*$ is

$$J_2\ddot{\alpha} = M - k_2\,\mathrm{sign}\,\dot{\alpha}\,\Sigma G_i s_i \quad (3.2)$$

where $s_i$ is the distance from $O^*$ to the $i$-th point of contact, taken with a plus sign if the point is on the same side of $O^*$ as the point $C_2$, and a minus sign otherwise.

The normal reactions satisfy the following equations

$$\Sigma G_i = m_2 g, \quad \Sigma G_i s_i = m_2 g a_2$$

Substituting these into Eqs (3.1) and (3.2), we obtain

$$N = m_2 a_2 \dot{\alpha}^2, \quad R = m_2 (a_2 \ddot{\alpha} + g k_2 \operatorname{sign} \dot{\alpha})$$
$$M = J_2 \ddot{\alpha} + m_2 g k_2 a_2 \operatorname{sign} \dot{\alpha} \tag{3.3}$$

We now introduce notation for the maximum values of the angular velocity and angular acceleration in slow motions

$$\omega_0 = \max |\dot{\alpha}|, \quad \varepsilon_0 = \max |\ddot{\alpha}| \tag{3.4}$$

The maximum is taken over the entire slow motion.

Formulae (3.3) and (3.4) imply the following estimates

$$|N| \leq m_2 a_2 \omega_0^2, \quad |R| \leq m_2 a_2 (\varepsilon_0 + g k_2 a_2^{-1}), \quad |M| \leq J_2 \varepsilon_0 + m_2 g k_2 a_2 \tag{3.5}$$

Let us consider equilibrium of the system consisting of the body and the mass $m_0$ at the point $O^*$. In the $Oxy$ plane this system is acted upon by forces $(-N)$ and $(-R)$ applied at the point $O^*$, a torque $(-M)$ and friction forces. The system will be in equilibrium if, at any point $Q$ in the $Oxy$ plane, the magnitude of the torque $M_1$ about that point of all forces except the friction forces acting on the system does not exceed the maximum possible magnitude $M_f$ of frictional forces (according to Coulomb's law) relative to that point. This condition means that, if $Q$ is considered as the instantaneous centre of velocities for the system, then the applied forces will not produce rotation of the system about that point, since the torque of the friction forces can counteract the applied torques. Note that, according to this formulation of the equilibrium conditions, not only rotational but also linear motions are impossible; in the latter case the point $Q$ must tend to infinity.

We introduce a Cartesian system of coordinates $O^*\xi\eta$ whose axis $O^*\xi$ is directed along the segment $O^*C_1$ and its axis $O^*\eta$ is perpendicular to the latter. The coordinates of the point $Q$ in the system $O^*\xi\eta$ will be denoted by $\xi$, $\eta$.

A necessary and sufficient condition for the body to be stationary is the following inequality.

$$|M_1| \leq M_f, \quad \forall \xi, \eta \tag{3.6}$$

which holds for all $\xi$, $\eta$, including their infinite values.

The torque $M_1$ is

$$M_1 = -M + \xi(N \sin\alpha - R\cos\alpha) - \eta(N\cos\alpha + R\sin\alpha) \tag{3.7}$$

Using the Cauchy inequality and estimates (3.5), we deduce from (3.7)

$$|M_1| \leq |M| + (N^2 + R^2)^{1/2}(\xi^2 + \eta^2)^{1/2} \leq J_2\varepsilon_0 + m_2 g k_2 a_2 +$$
$$+ m_2 a_2 [\omega_0^4 + (\varepsilon_0 + g k_2 a_2^{-1})^2]^{1/2} r \tag{3.8}$$

where we have introduced the notation

$$r = (\xi^2 + \eta^2)^{1/2} = O^*Q \tag{3.9}$$

Let us evaluate the magnitude $M_f$ of the maximum possible torque about $Q$ of the friction forces acting on the body and the mass $m_0$. This value $M_f$ will be obtained if the friction force at each point of contact is of maximum magnitude and directed perpendicular to the segment connecting $Q$ to that point. We obtain

$$M_f = k_1 \Sigma P_i r_i + k_0 m_0 g r, \quad r_i = [(\xi - \xi_i)^2 + (\eta - \eta_i)^2]^{1/2} \tag{3.10}$$

where $P_i$ is the normal reaction at a certain point of contact of the body with the $Oxy$ plane, $(\xi_i, \eta_i)$ are the coordinates of that point of contact in the system $O^*\xi\eta$, $r_i$ is the distance from that point to $Q$, and $r$ is defined by (3.9). Summation over $i$ in (3.10) and subsequent formulae ranges over all points of contact of the body with the $Oxy$ plane, summation being replaced by integration if contact occurs over entire regions.

The normal reactions $P_i$ satisfy the following equations and inequalities

$$\Sigma P_i = m_1 g, \quad \Sigma P_i \xi_i = m_1 g a_1, \quad \Sigma P_i \eta_i = 0, \quad P_i > 0 \tag{3.11}$$

Equations (3.11) reflect the fact that the mass of the body is $m_1$ and its mass centre has coordinates $(a_1, 0)$ in the $O^*\xi\eta$ system. If the body is in contact with the plane at more than three points, Eqs (3.11) do not uniquely determine the normal reactions $P_i$ (static indeterminacy).

In order to derive the sufficient conditions for the body to be motionless, we will determine a lower bound for $M_f$ which holds for any $\xi$, $\eta$ and any distribution of the normal reactions satisfying Eqs (3.11), and then require this bound to be at least equal to the upper bound (3.8) for $|M_1|$. The relations (3.10) and (3.11) imply the following estimates

$$\begin{aligned} M_f &\geqslant k_1 \Sigma P_i |\xi - \xi_i| + k_0 m_0 g r \geqslant k_1 |\xi \Sigma P_i - \Sigma P_i \xi_i| + \\ &+ k_0 m_0 g r = k_1 m_1 g |\xi - a_1| + k_0 m_0 g r \end{aligned} \tag{3.12}$$

Using estimates (3.8) and (3.12), we obtain the sufficient condition for the body to remain motionless, as an inequality

$$b_0 \leqslant b_1 |\xi - a_1| + b_2 r \tag{3.13}$$

where we have introduced the notation

$$\begin{aligned} &b_0 = J_2 \varepsilon_0 + m_2 g k_2 a_2, \quad b_1 = m_1 g k_1, \\ &b_2 = m_0 g k_0 - m_2 a_2 [\omega_0^4 + (\varepsilon + g k_2 a_2^{-1})^2]^{1/2} \end{aligned} \tag{3.14}$$

Inequality (3.13) must hold for all $\xi$, $\eta$. We first put $\xi = \eta = 0$ in (3.13), and then $\xi = a_1$, $\eta = 0$. Using equality (3.9), we obtain

$$b_0 \leqslant b_1 a_1, \quad b_0 \leqslant b_2 a_1 \tag{3.15}$$

Suppose both inequalities (3.15) hold. Then $b_2 > 0$ and we have the bound

$$b_1 |\xi - a_1| + b_2 r \geqslant b_1 |\xi - a_1| + b_2 |\xi|$$

The right-hand side of the last inequality is a piecewise-linear function of $\xi$, which has a maximum at either $\xi = a_1$ or $\eta = 0$. Consequently, by (3.15), we have

$$b_1 |\xi - a_1| + b_2 r \geqslant \min(b_1 a_1, b_2 a_1) \geqslant b_0$$

Thus, if both inequalities (3.15) hold, inequalities (3.13) hold for all $\xi$, $\eta$. Consequently, the joint validity of inequalities (3.15) is the sufficient condition for the body to be motionless.

In developed form, taking notation (3.14) into account, the sufficient conditions for the body to be motionless are

$$\begin{aligned} &J_2 \varepsilon_0 + m_2 g k_2 a_2 \leqslant m_1 g k_1 a_1 \\ &J_2 \varepsilon_0 + m_2 g k_2 a_2 + m_2 a_1 a_2 [\omega_0^4 + (\varepsilon + g k_2 a_2^{-1})^2]^{1/2} \leqslant m_0 g k_0 a_1 \end{aligned} \tag{3.16}$$

Suppose the slow motions occur at fairly low angular velocities and accelerations, so that $\omega_0$ and $\varepsilon_0$ are very small. Then conditions (3.16) become

$$m_2 k_2 a_2 < m_1 k_1 a_1, \quad m_2 k_2 (a_1 + a_2) < m_0 k_0 a_1 \tag{3.17}$$

If inequalities (3.17) are satisfied, one can always ensure that the body will be motionless by sufficiently slow rotation of the tail (with small $\omega_0$ and $\varepsilon_0$).

## 4. ANALYSIS OF FAST MOTIONS

We will use the conservation laws (2.2). To do this, we first differentiate relations (1.2) with respect to $t$:

$$\dot{x}_1 = \dot{x}_0 - \dot{\theta} a_1 \sin\theta, \quad \dot{y}_1 = \dot{y}_0 + \dot{\theta} a_1 \cos\theta \tag{4.1}$$

$$\dot{x}_2 = \dot{x}_0 + (\dot{\theta} + \dot{\alpha}) a_2 \sin(\theta + \alpha), \quad \dot{y}_2 = \dot{y}_0 - (\dot{\theta} + \dot{\alpha}) a_2 \cos(\theta + \alpha)$$

We then differentiate relations (1.3), after first substituting (4.1) into them and using the fact that, by (2.2), $\dot{x}_c = \dot{y}_c = 0$. We obtain two relations, from which we obtain the derivatives

$$\dot{x}_0 = m^{-1}[m_1 a_1 \dot{\theta} \sin\theta - m_2 a_2 (\dot{\theta} + \dot{\alpha}) \sin(\theta + \alpha)]$$

$$\dot{y}_0 = m^{-1}[-m_1 a_1 \dot{\theta} \cos\theta + m_2 a_2 (\dot{\theta} + \dot{\alpha}) \cos(\theta + \alpha)]$$

Now, substituting these derivatives into formula (1.6) for $K$ and bearing in mind that, by (2.2), $K = 0$, we obtain a linear homogeneous relation for the derivatives $\dot{\theta}$ and $\dot{\alpha}$, which, after some algebra, reduces to the following from:

$$d\theta / d\alpha = -\varphi(\alpha) \tag{4.2}$$

$$\varphi(\alpha) = \frac{mJ_2 - m_2^2 a_2^2 + m_1 m_2 a_1 a_2 \cos\alpha}{m(J_1 + J_2) - m_1^2 a_1^2 - m_2^2 a_2^2 + 2 m_1 m_2 a_1 a_2 \cos\alpha}$$

It follows from (4.2) that the variation $\Delta\theta$ of the angle $\theta$ during the fast motion depends only on the initial and final values $\alpha^0$ and $\alpha^1$ of the angle $\alpha$ in that motion, but not on the law governing the variation of the angle $\alpha$. We have

$$\Delta\theta = -\int_{\alpha^0}^{\alpha^1} \varphi(\alpha) d\alpha = \gamma(\alpha_0) - \gamma(\alpha_1), \quad \gamma(\beta) = \int_0^\beta \varphi(\alpha) d\alpha \tag{4.3}$$

Evaluating the integral $\gamma(\beta)$ we obtain [8]

$$\gamma(\beta) = \frac{\beta}{2} + \frac{A_0}{A_+ A_-} \operatorname{arctg}\left(\frac{A_+}{A_-} \operatorname{tg}\frac{\beta}{2}\right) \tag{4.4}$$

$$A_0 = m(J_2 - J_1) + m_1^2 a_1^2 - m_2^2 a_2^2,$$

$$A_\pm = [m(J_1 + J_2) - (m_1 a_1 \pm m_2 a_2)^2]^{1/2}$$

As $\beta$ varies from 0 to $\pi$, the function $\gamma(\beta)$ increases monotonically from 0 to

$$\gamma(\pi) = (\pi/2)(A_0 + A_+ A_-)(A_+ A_-)^{-1}$$

Now, integrating relations (4.2), we can calculate the increments of the coordinates of the hinge $O^*$ during the fast motion

$$\Delta x_0 = m^{-1}[-m_1 a_1 \Delta\cos\theta + m_2 a_2 \Delta\cos(\theta + \alpha)] \tag{4.5}$$

$$\Delta y_0 = m^{-1}[-m_1 a_1 \Delta\sin\theta + m_2 a_2 \Delta\sin(\theta + \alpha)]$$

## 5. LONGITUDINAL DISPLACEMENT OF THE TWO-LINK MECHANISM

We will now describe a sequence of elementary motions which leads to longitudinal displacement of the two-link mechanism. Suppose that at the starting time the mechanism is at rest, constituting a segment parallel to the $x$ axis (state 0 in Fig. 2). We have $\theta = \alpha = 0$ in state 0. In addition, we will assume, without loss of generality, that $x_0 = y_0 = 0$ in that state.

1. Perform a slow motion in which the tail turns through an angle $\beta$ and the body remains motionless. The mechanism will reach state 1 in Fig. 2, in which

$$\theta = 0, \quad \alpha = \beta, \quad x_0 = y_0 = 0$$

2. Perform a fast motion in which the angle $\alpha$ varies from $\beta$ to 0. The mechanism reaches state 2 in Fig. 2. In this state, by formulae (4.3) and (4.5), we have

$$\theta = \gamma(\beta), \quad x_0 = m^{-1}[m_1a_1(1-\cos\gamma) + m_2a_2(\cos\gamma - \cos\beta)] \tag{5.1}$$

$$y_0 = m^{-1}[-m_1a_1\sin\gamma + m_2a_2(\sin\gamma - \sin\beta)]$$

Figure 2 illustrates only the variations of the angle $\alpha$, but those of the angle $\theta$ and the coordinates $x_0, y_0$ are not shown.

3. By means of a slow motion, change the angle $\alpha$ from 0 to $-\beta$. The mechanism reaches state 3 in Fig. 2. The angle $\theta$ and the coordinates $x_0, y_0$ remain as before and are given by formulae (5.1).

4. By means of a fast motion, change the angle $\alpha$ from $-\beta$ to 0. The mechanism reaches state 4 in Fig. 2. In that state, by formulae (4.3) and (4.5), we have

$$\theta = 0, \quad x_0 = m^{-1}m_2a_2[\cos\gamma - \cos\beta + 1 - \cos(\gamma - \beta)] \tag{5.2}$$

$$y_0 = m^{-1}m_2a_2[\sin\gamma - \sin\beta - \sin(\gamma - \beta)]$$

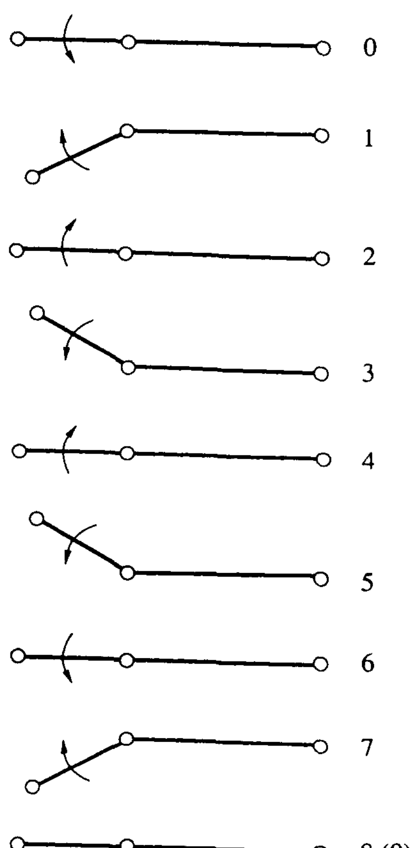


Fig. 2

As a result of these motions the two-link mechanism has again become a segment parallel to the $x$ axis, but it has been displaced laterally ($y_0 \neq 0$). To cancel out this displacement, repeat the above motions in a different order, namely, perform motions 3, 4, 1, 2.

5. Using a slow motion, change the angle $\alpha$ from 0 to $-\beta$. The mechanism reaches state 5 in Fig. 2, in which the variables $\theta, x_0, y_0$ are given by formulae (5.2).

6. Using a fast motion, change the angle $\alpha$ from $-\beta$ to 0. The mechanism reaches state 6 in Fig. 2, in which, according to formulae (4.3), (4.5) and (5.2), we obtain

$$x_0 = m^{-1}\{m_1 a_1(1-\cos\gamma) + m_2 a_2[2\cos\gamma - 2\cos\beta + 1 - \cos(\gamma-\beta)]\} \tag{5.3}$$
$$\theta = -\gamma, \quad y_0 = m^{-1}[-m_1 a_1 \sin\gamma + m_2 a_2 \sin(\gamma-\beta)]$$

7. Using a slow motion, change the angle $\alpha$ from 0 to $\beta$. The mechanism reaches state 7 in Fig. 2, in which the variable $\theta, x_0, y_0$ are given by formulae (5.3).

8. Using a fast motion, change the angle $\alpha$ from $\beta$ to 0. The mechanism will reach state 8 in Fig. 2. By formulae (4.3), (4.5) and (5.3), we obtain in that state

$$\theta = 0, \quad x_0 = 2m^{-1} m_2 a_2[\cos\gamma - \cos\beta + 1 - \cos(\gamma-\beta)], \quad y_0 = 0 \tag{5.4}$$

State 8 differs from state 0 in Fig. 2 only in that the mechanism has moved along the $x$ axis by a distance given by formulae (5.4). This displacement may be represented in the form

$$l = 8m^{-1} m_2 a_2 \sin(\beta/2) \times \cos(\gamma/2)\sin[(\beta-\gamma)/2] \tag{5.5}$$

where the function $\gamma(\beta)$ is defined by formula (4.4).

The cycle of four slow and four fast motions that takes the two-link mechanism from state 1 to state 8 in Fig. 2 may be repeated any number of times $n$. By a suitable choice of the number $n$ and of the angle $\beta$ through which the tail turns, one can achieve any longitudinal displacement of the mechanism.

The average velocity of the longitudinal motions is

$$v = l[4(T+\tau)]^{-1} \quad (T \gg \tau) \tag{5.6}$$

where $T$ and $\tau$ are the durations of the slow and fast motions, respectively.

## 6. ARBITRARY DISPLACEMENT OF THE TWO-LINK MECHANISM

We will now show how elementary motions may be used to achieve any desired displacement of the mechanism in the $Oxy$ plane. Suppose that in the initial and final states the mechanism is aligned along a straight line and at rest. Without loss of generality, we shall assume that in the initial state

$$x_0 = y_0 = 0, \quad \theta = \alpha = 0$$

The final state will be written in the form

$$x_0 = x^*, \quad y_0 = y^*, \quad \theta = \theta^*, \quad \alpha = 0 \tag{6.1}$$

The desired displacement will be constructed from the following steps.

1. By means of a slow motion, change the angle $\alpha$ from 0 to $\beta_1$.

2. By means of a fast motion, change the angle $\alpha$ from $\beta_1$ to $-\beta_1$. As a result, according to formulae (4.3), (4.5), the angle $\theta$ and the coordinates $x_0, y_0$ take the following values

$$\theta = \theta_1 = 2\gamma(\beta_1) \tag{6.2}$$
$$x_0 = m^{-1}\{m_1 a_1(1-\cos\theta_1) + m_2 a_2[\cos(\theta_1-\beta_1) - \cos\beta_1]\}$$
$$y_0 = m^{-1}\{-m_1 a_1 \sin\theta_1 + m_2 a_2[\sin(\theta_1-\beta_1) - \sin\beta_1]\}$$

3. By means of a slow motion, change the angle $\alpha$ from $-\beta_1$ to $\beta_1$.

4. Perform the sequence of motions 2 and 3 $n_1$ times and then, by a slow motion, change the angle $\alpha$ from $\beta_1$ to 0. The two-link mechanism will again assume a linear shape, and by (6.2) its position and orientation will be defined by the formulae

$$x_0 = x' = \Sigma_{x1}, \quad y_0 = y' = \Sigma_{y1}, \quad \theta = \theta' = n_1\theta_1, \quad \theta_1 = 2\gamma(\beta_1) \tag{6.3}$$

where

$$\Sigma_{xi} = m^{-1}\{m_1 a_1[1-\cos(n_i\theta_i)] + m_2 a_2 \sum_{j=1}^{n_i} [\cos(j\theta_i - \beta_i) - \cos(j\theta_i - \theta_i + \beta_i)]\}$$

$$\Sigma_{yi} = m^{-1}\{-m_1 a_1 \sin(n_i\theta_i) + m_2 a_2 \sum_{j=1}^{n_i} [\sin(j\theta_i - \beta_i) - \sin(j\theta_i - \theta_i + \beta_i)]\}$$

5. Perform the linear displacement of the mechanism described in Section 5 with some parameter $\beta$. After $n$ cycles of this displacement, the coordinates of the hinge $O^*$ will be

$$x_0 = x'' = x' + nl\cos\theta', \quad y_0 = y'' = y' + nl\sin\theta' \tag{6.4}$$

while the angle $\theta$ will remain equal to $\theta'$. The displacement $l$ is defined by (5.5), and $x'$, $y'$, $\theta'$ by Eqs (6.3). The two-link mechanism is still straight.

6. Perform the sequence of motions 1–4 with the angle $\beta_1$ replaced by $\beta_2$ and the number $n_1$ by $n_2$. As a result, the mechanism will again be straight in form. Its new position and orientation will be defined by formulae analogous to (6.3):

$$x_0 = x'' + \Sigma_{x2}, \quad y_0 = y'' + \Sigma_{y2}, \quad \theta = \theta' + n_2\theta_2, \quad \theta_2 = 2\gamma(\beta_2) \tag{6.5}$$

The variables $x_0$, $y_0$ and $\theta$ at the end of the motion, as defined by (6.5), must equal the given values (6.1). Substituting expressions (6.5) into formulae (6.1), as well as (6.3) and (6.4), we calculate the trigonometric sums in (6.3) and (6.5) [8].

After some reduction we obtain the equations

$$\begin{aligned} &\theta' + \theta'' = \theta^* \\ &B_1(1-\cos\theta') + B_2(1-\cos\theta'') + nl\cos\theta' = x^* \\ &-B_1\sin\theta' - B_2\sin\theta'' + nl\sin\theta' = y^* \end{aligned} \tag{6.6}$$

where we have put

$$\begin{aligned} &\theta' = n_1\theta_1, \quad \theta'' = n_2\theta_2, \quad \theta_i = 2\gamma(\beta_i), \quad i = 1,2 \\ &B_i = m^{-1}\{m_1 a_1 + m_2 a_2 \sin(\beta_i - \theta_i/2)[\sin(\theta_i/2)]^{-1}\} \end{aligned} \tag{6.7}$$

If we substitute expressions (6.7) into formulae (6.6), we obtain three transcendental equations for the three unknown integers $n_1$, $n_2$, $n$ and the three angles $\beta_1$, $\beta_2$, $\beta$ on which the angles $\theta_1$, $\theta_2$ and the displacement $l$, respectively, depend (see (6.7) and (5.5)).

We will show that these equations are solvable in the important case in which the final point is quite far from the initial point, i.e., with the condition

$$\max(a_1, a_2)/d = \varepsilon \ll 1, \quad d = [(x^*)^2 + (y^*)^2]^{1/2} \tag{6.8}$$

Let us assume that the angles $\beta_1$, $\beta_2$ and $\beta$ are so small and the numbers $n_1$, $n_2$ and $n$ so large that $\theta_1$, $\theta_2$ and $l$ are also small and the products $n_1\theta_1$, $n_2\theta_2$ and $nl$ are finite. It follows from (4.4) that for small angles $\beta$

$$\gamma(\beta) = A\beta, \quad A = (1 + A_0 A_-^{-2})/2 < 1 \tag{6.9}$$

We then have, by formulae (5.5) and (6.7)

$$l = D\beta^2, \quad D = 2m^{-1}m_2a_2(1-A), \quad \theta_i = 2A\beta_i \tag{6.10}$$

$$B_i = b = m^{-1}[m_1a_1 + m_2a_2(A^{-1}-1)], \quad i = 1,2$$

Substituting expressions (6.10) into (6.6), we obtain

$$\theta' + \theta'' = \theta^*, \quad b(2 - \cos\theta' - \cos\theta'') + L\cos\theta' = x^* \tag{6.11}$$

$$-b(\sin\theta' + \sin\theta'') + L\sin\theta' = y^*, \quad L = nl$$

The first equation of (6.11) gives $\theta'' = \theta^* - \theta'$, and we then eliminate $L$ from the second and third equations of (6.11). This gives a transcendental equation for $\theta'$:

$$\operatorname{tg}\theta' = \frac{y^* + b[\sin\theta' + \sin(\theta^* - \theta')]}{x^* - b[2 - \cos\theta' - \cos(\theta^* - \theta')]} \tag{6.12}$$

It follows from (6.10) and (6.8) that

$$b \sim \max(a_1, a_2) \sim \varepsilon d$$

Since $\varepsilon \ll 1$, a solution of Eq. (6.12) will be sought in the form

$$\theta' = \theta_0' + \theta_1', \quad \theta_0' = \operatorname{arctg}(y^* / x^*) \tag{6.13}$$

where $\theta_1'$ is a small quantity of the order of $\varepsilon$. Substituting $\theta'$ from (6.13) into Eq. (6.12) and simplifying, assuming that $b \sim \varepsilon d$, $\theta_1' \sim \varepsilon$, we obtain

$$\theta_1' = bd^{-1}[2\sin\theta_0' + \sin(\theta^* - 2\theta_0')] \tag{6.14}$$

Equations (6.13) and (6.14) determine the solution $\theta'$ of Eq. (6.12) with an error of the order of $\varepsilon^2$. Using formulae (6.11), we obtain, with the same degree of accuracy

$$\theta'' = \theta^* - \theta', \quad L = d + b[1 - 2\cos\theta_0' + \cos(\theta^* - 2\theta_0')] \tag{6.15}$$

Equations (6.13)–(6.15) determine the required solution $\theta'$, $\theta''$, $L$ of system (6.11). On the other hand, it follows from (6.7), (6.10) and (6.11) that

$$\theta' = 2An_1\beta_1, \quad \theta'' = 2An_2\beta_2, \quad L = Dn\beta^2 \tag{6.16}$$

The constants $A$ and $D$ are given by Eqs (6.9) and (6.10). Choosing sufficiently small values of the angles $\beta_1$, $\beta_2$ and $\beta$ and sufficiently large integers $n_1$, $n_2$ and $n$, we can approximate the solution $\theta'$, $\theta''$, and $L$ given by (6.16) to any desired accuracy. It turns out that $n_i \sim \beta_i^{-1}$, $i = 1, 2$ and $n \sim \beta^{-2}$ (see (6.16)).

We have thus established that, provided condition (6.8) is satisfied, the two-link mechanism can be moved from any initial state to any final state (6.1) to any prescribed accuracy. We have demonstrated a constructive mode of control which carries out that displacement. Of course, this mode of control is far from unique; it was chosen so that the computation could be carried through to completion in explicit form. One can indicate modes of displacement requiring less time; their computation requires the solution of a system of transcendental equations.

The motions we have constructed consist of slow and fast motions. For fast motions to be possible, conditions (2.1) must hold, while the sufficient conditions for slow motions to be possible are given by inequalities (3.16). If the slow motions are sufficiently slow, conditions (3.17) may be applied.

## 7. SPECIAL CASES

We shall consider two simple and important special cases of a two-link mechanism. Let the body and tail be point masses of mass $m_1$ and $m_2$, respectively, attached to a hinge of mass $m_0$ by weightless rigid rods. In that case, in formulae (4.4) and (3.16) we must put

$$J_1 = m_1 a_1^2, \quad J_2 = m_2 a_2^2 \tag{7.1}$$

The second case is that of a mechanism consisting of two uniform straight rods of identical linear density $\rho$ attached to a hinge of negligible mass. The rods are in contact with the plane along their entire length, and each element of either rod is subject to a normal reaction and friction force proportional to the length of the element. The coefficient of friction of any element is assumed to be $k$. This is essentially a schematic model of a snake bent at one point.

Denoting the lengths of the body and tail by $l_1$ and $l_2$, respectively, and using the notation adopted above, we obtain

$$m_i = \rho l_i, \quad a_i = l_i/2, \quad J_i = \rho l_i^3/3, \quad i = 1,2 \tag{7.2}$$
$$m_0 = 0, \quad m = m_1 + m_2$$

Substituting expressions (7.2) into (4.4), we specify and simplify the formulae for $A_0$ and $A_\pm$ for this case

$$A_0 = (l_2^2 - l_1^2)(l_1^2 + l_2^2 + 4l_1 l_2) \tag{7.3}$$
$$A_+ = [(l_1 + l_2)^4 - 12 l_1^2 l_2^2]^{1/2}, \quad A_- = (l_1 + l_2)^2$$

All the relations for fast motions remain valid; we need only substitute Eqs (7.2) and (7.3) into them.

The situation is different with regard to the conditions for slow motions to be feasible; inequalities (3.16) and (3.17) do not hold for $m_0 = 0$. To show that slow motions are possible for the model under consideration, more detailed estimates are required.

Taking (7.2) into account, inequality (3.8) becomes

$$|M_1| \leqslant \rho l_2^2 \{l_2 \varepsilon_0/3 + gk/2 + [\omega_0^4 + (\varepsilon_0 + 2gkl_2^{-1})^2]^{1/2} r/2\} \tag{7.4}$$

Formula (3.10) for the magnitude of the torque produced by the friction forces in this case takes the form

$$M_f = k\rho g I, \quad I = \int_0^{l_1} [(\xi - s)^2 + \eta^2]^{1/2} ds \tag{7.5}$$

We will estimate the integral $I$ from below in two ways. Using the obvious inequality $|a| + |b| \leqslant (2a^2 + 2b^2)^{1/2}$, we obtain

$$I \geqslant (I_0 + |\eta| l_1)/\sqrt{2}$$
$$I_0 = \int_0^{l_1} |\xi - s| ds = \begin{cases} |\xi| l_1 + l_1^2/2, & \xi < 0 \\ l_1^2/2 + \xi(\xi - l_1), & 0 \leqslant \xi \leqslant l_1 \\ \xi l_1 - l_1^2/2, & \xi > l_1 \end{cases} \tag{7.6}$$

From the obvious inequality

$$l_1^2/2 + \xi(\xi - l_1) \geqslant (\sqrt{2} - 1) l_1 \xi$$

we obtain the bound

$$I_0 \geqslant (\sqrt{2} - 1)|\xi| l_1 \tag{7.7}$$

which is easily seen to hold for all three cases of (7.6), i.e. for any $\xi$. From formulae (7.6) and (7.7) we obtain the bound

$$I \geqslant (1 - 1/\sqrt{2})(|\xi| + |\eta|) l_1 \geqslant (1 - 1/\sqrt{2}) l_1 r, \quad r = (\xi^2 + \eta^2)^{1/2} \tag{7.8}$$

On the other hand, by (7.5) and (7.6), we have

$$I \geqslant I_0 \geqslant l_1^2/4 \tag{7.9}$$

As a result, using formulae (7.5), (7.8) and (7.9), we have the limit bound

$$M_f \geqslant (k\rho g l_1/4)\max(l_1, 2(2-\sqrt{2})r) \tag{7.10}$$

Using bounds (7.4) for $|M_1|$ and (7.10) for $M_f$, we see that the sufficient condition for (3.6) to hold is that, for all $r \geqslant 0$

$$c_1 + c_2 r \leqslant c_3 \max(c_0 l_1, r) \tag{7.11}$$

where we have put

$$c_1 = (\tfrac{2}{3})\varepsilon_0 l_2^3 + gkl_2^2, \quad c_2 = [\omega_0^4 + (\varepsilon_0 + 2gkl_2^{-1})^2]^{1/2} l_2^2 \tag{7.12}$$
$$c_3 = (2-\sqrt{2})gkl_1, \quad c_0 = (2+\sqrt{2})/4$$

The expression on the left-hand side of inequality (7.11) is an increasing linear function of $r$, while that on the right-hand side is a piecewise-linear convex function of $r$ with a jump of derivative at $r = c_0 l_1$. For inequality (7.11) to hold for all $r \geqslant 0$, it is necessary and sufficient that it holds at $r_0 = c_0 l_1$ and as $r \to \infty$. We obtain the conditions

$$c_1 + c_2 c_0 l_1 \leqslant c_3 c_0 l_1, \quad c_2 \leqslant c_3 \tag{7.13}$$

It is easy to see that the second condition of (7.13) follows from the first. Consequently, the first inequality is the sufficient condition for a slow motion to be possible in this model. Written out in full, taking the notation (7.12) into account, this condition takes the form

$$(\tfrac{2}{3})\varepsilon_0 l_2^3 + gkl_2^2 + (2+\sqrt{2})[\omega_0^4 + (\varepsilon_0 + 2gkl_2^{-1})^2]^{1/2} l_1 l_2^2/4 \leqslant gkl_1^2/2 \tag{7.14}$$

At vanishingly small angular velocities and accelerations $\omega_0$ and $\varepsilon_0$, inequality (7.14) becomes

$$2\lambda^2 + (2+\sqrt{2})\lambda - 1 \leqslant 0, \quad \lambda = l_2/l_1 \tag{7.15}$$

Since $\lambda > 0$, this inequality will hold if

$$\lambda = l_2/l_1 \leqslant \lambda_0 = [(14+4\sqrt{2})^{1/2} - 2 - \sqrt{2}]/4 = 0.255 \tag{7.16}$$

If condition (7.16) is satisfied, the sufficient condition (7.14) for the body to be motionless may always be guaranteed by choosing sufficiently small $\omega_0$ and $\varepsilon_0$, that is, provided the rotations of the tail are sufficiently slow.

## 8. EXAMPLES

Let us assume that the slow motions consist of accelerating and decelerating phases with angular velocities of the same magnitude $\varepsilon_0$. Then the magnitude of the angular velocity $\omega(t) = |\dot\alpha(t)|$ varies as

$$\omega(t) = \varepsilon_0 t, \quad t \in [0, T/2]; \quad \omega(t) = \varepsilon_0(T-t), \quad t \in [T/2, T] \tag{8.1}$$

and the angle $\beta$ through which the tail turns in the slow motions and maximum magnitude of the angular velocity $\omega_0$ are

$$\beta = \varepsilon_0 T^2/4, \quad \omega_0 = \varepsilon_0 T/2 \tag{8.2}$$

Suppose that the prescribed parameters (8.1) and (8.2) of the motion are

$$\varepsilon_0 = 4\ \mathrm{s}^{-2}, \quad T = 1\ \mathrm{s}, \quad \beta = 1\ \mathrm{rad}, \quad \omega_0 = 2\ \mathrm{s}^{-1} \tag{8.3}$$

Consider two versions of a two-link mechanism moving with characteristics (8.1)–(8.3).

1. The mechanism consists of two point masses and has the following parameters:

$$m_0 = 0.6 \text{ kg}, \quad m_1 = 0.3 \text{ kg}, \quad m_2 = 0.3 \text{ kg}, \quad m = 1.2 \text{ kg}$$

$$a_1 = 1 \text{ m}, \quad a_2 = 0.2 \text{ m}, \quad k_0 = k_1 = k_2 = 0.2$$

A check of conditions (3.16), taking (7.1) into account, shows that they are satisfied. The longitudinal displacement per cycle, evaluated using formulae (5.5) and (4.4), turns out to be $l$ = 0.085 m, and the average velocity of longitudinal motion, according to formula (5.6), is $\upsilon$ = 0.021 m $s^{-1}$.

2. The mechanism consists of two uniform rods and has the same mass and dimensions as in the first example.

$$m_1 = 1 \text{ kg}, \quad m_2 = 0.2 \text{ kg}, \quad m = 1.2 \text{ kg}$$

$$l_1 = 1 \text{ m}, \quad l_2 = 0.2 \text{ m}, \quad \rho = 1 \text{ kg m}^{-1}, \quad k = 0.2$$

A check shows that condition (7.14) for the motions to be possible is satisfied. The longitudinal displacement $l$ is evaluated by formula (5.5), taking formulae (4.4), (7.2) and (7.3) into account. We obtain

$$l = 0.028 \text{ m}$$

The average velocity of longitudinal motion, according to formula (5.6), is

$$\upsilon_0 = 0.007 \quad \text{m s}^{-1}$$

By formulae (2.1), the magnitude of the controlling torque necessary to implement the motions in both examples is of the order of 8 N. m.

## 9. CONCLUSION

The above investigation shows that a two-link mechanism can be moved over a rough horizontal surface. The mechanism can move along a straight line as a whole, rotate and implement any displacement in the plane. Sufficient conditions for this mode of motion to be possible have been established. The motion is controlled by a single motor mounted at the hinge of the mechanism. The most natural way of doing this is to use an electric motor. This mode of motion is remarkable for the simplicity of both the structure of the device and the mode of control. Its use for some types of mobile robots seems promising.

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